CDI-II — Prática F.4 23/3/21

$$\boxed{\text{Ficha 4}}$$

Teorema da função composta

$$\mathbb{R}^n \xrightarrow[\text{dif. em } a]{g} \mathbb{R}^p \xrightarrow[\text{dif. em } g(a)]{f} \mathbb{R}^m$$

$$\begin{tikzcd}
a \arrow[r, mapsto] \arrow[rr, bend right, "f\circ g \text{ dif. em } a"'] & g(a) \arrow[r, mapsto] & f(g(a))
\end{tikzcd}$$

$$\underbrace{D(f\circ g)(a)}_{m\times n} = \underbrace{Df(g(a))}_{m\times p}\,\underbrace{Dg(a)}_{p\times n}$$

1 — Trivial ✓ $g(1,1) = (e^0, 0) = (1,0)$

$$D(f\circ g)(1,1) = Df(g(1,1))\, Dg(1,1)$$

3×2

$$= Df(1,0)\, Dg(1,1)$$

$\quad\quad 3\times 2 \quad\quad 2\times 2$

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2 — σ = "sigma" , γ = "gama"

$$\sigma(t) = F(\gamma(t)) = (F\circ\gamma)(t)$$

$$\begin{tikzcd}
t \arrow[r, mapsto] & \gamma(t) \arrow[r, mapsto] & F(\gamma(t)) \\
\mathbb{R} \arrow[r] \arrow[rr, bend right, "\sigma = F\circ\gamma"'] & \mathbb{R}^3 \arrow[r] & \mathbb{R}
\end{tikzcd}$$

$$\sigma'(t) = DF(\gamma(t))\,\gamma'(t)$$

$$\sigma'(t) = \nabla F(\gamma(t))\,\gamma'(t)$$

$$\begin{bmatrix} \cdots \end{bmatrix} \begin{bmatrix} \vdots \end{bmatrix}$$ matrizes

$$\sigma'(t) = \nabla F(\gamma(t)) \cdot \gamma'(t)$$ vetores

$$\nabla F(x,y,z) = (2x, 2y, 2z) \text{ ou}$$

$$= \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$$

$$\gamma(t) = (\underbrace{\operatorname{sen} t}_{x}, \underbrace{t^2}_{y}, \underbrace{\cos t}_{z})$$

$$\nabla f(\gamma(t)) = (2\sin t,\ 2t^2,\ 2\cos t)$$

$$\gamma'(t) = \begin{bmatrix} \cos t \\ 2t \\ -\sin t \end{bmatrix} = (\cos t, 2t, -\sin t)$$

$$\sigma'(t) = (2\sin t,\ 2t^2,\ 2\cos t) \cdot (\cos t, 2t, -\sin t)$$

ou

$$\begin{bmatrix} 2\sin t & 2t^2 & 2\cos t \end{bmatrix} \begin{bmatrix} \cos t \\ 2t \\ -\sin t \end{bmatrix}$$

$$\boxed{\sigma'(t) = 4t^3}$$

Alternativamente, usar a regra da cadeia:

$$\sigma(t) = F(\gamma(t)) = F(x(t), y(t), z(t))$$

$$\begin{cases} x(t) = \operatorname{sen} t \\ y(t) = t^2 \\ z(t) = \cos t \end{cases}$$

Cadeias: $F\,x\,t$, $F\,y\,t$, $F\,z\,t$

$\sigma\ t$

$$\sigma'(t) = \underbrace{\frac{\partial F}{\partial x}}_{2x \,=\, 2\operatorname{sen} t}\underbrace{\frac{dx}{dt}}_{\cos t} + \underbrace{\frac{\partial F}{\partial y}}_{2y \,=\, 2t^2}\underbrace{\frac{dy}{dt}}_{2t} + \underbrace{\frac{\partial F}{\partial z}}_{2z \,=\, 2\cos t}\underbrace{\frac{dz}{dt}}_{-\operatorname{sen} t}$$

$$= 4t^3$$

$\gamma(t) = (x(t), y(t), z(t))$

$(\operatorname{sen} t, t^2, \cos t)$

t

$$\sigma(t) = F(\gamma(t)) \qquad \mathbb{R} \ni t \longmapsto \gamma(t) \in \mathbb{R}^3$$

$$\sigma(t) = F(x(t), y(t), z(t))$$

$$\sigma'(t) = \frac{d\sigma}{dt}(t) = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}$$

3- $D_v(f \circ g)(0,0) = \underbrace{D(f \circ g)(0,0)}\, v$

$$D(f \circ g)(0,0) = Df(g(0,0))\, Dg(0,0)$$

$$= \underbrace{Df(0,1,2)}_{\text{calcular}}\, \underbrace{Dg(0,0)}_{\text{dada}}$$

✓

4- $\sigma(x) = f(\operatorname{sen} x,\ x + e^{x})$

$$x \overset{g}{\longmapsto} (\operatorname{sen} x,\ x+e^{x}) \longmapsto f(\operatorname{sen} x,\ x+e^{x})$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r] \arrow[rr, bend right, "\sigma = f \circ g"'] & \mathbb{R}^2 \arrow[r, mapsto] & \mathbb{R}^3
\end{tikzcd}$$

$$g(x) = (\operatorname{sen} x,\ x + e^{x})$$

$$g: \mathbb{R} \longrightarrow \mathbb{R}^2, \quad C^1$$

$$\sigma'(0) = Df(g(0))\, Dg(0)$$

$$g(0) = (0, 1)$$

$$\sigma'(0) = \underset{3\times 2}{Df(0,1)}\,\underset{2\times 1}{Dg(0)}$$

3×1

$$= \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

etc.... contas

5- Regra da cadeia!

a)
$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & \mathbb{R}^3 \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

5-a) $\frac{\partial}{\partial y}(g\circ f)(1,1,0)$? $f(1,1,0)=(2,2,1)$

$$D(g\circ f)(1,1,0) = Dg(f(1,1,0))\,Df(1,1,0)$$

$\nabla g(2,2,1)$

1×3

$$\begin{bmatrix} \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$g(f(x,y,z)) = g(\underbrace{u(x,y,z)}_{x^2+y^2+z^2}, \underbrace{v(x,y,z)}_{x+y-z}, \underbrace{w(x,y,z)}_{xy\,e^z})$$

$$\frac{\partial}{\partial y}(g\circ f)(x,y,z) = \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial y}$$

$(2,2,1)$ $(1,1,0)$

5-b) trivial depois de 5-a)

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7 - $\boxed{F(x, y, \overset{z}{\overline{g(x,y)}}) = 0}$

$\Leftrightarrow$ $\boxed{z = g(x,y)}$

$\frac{\partial F}{\partial z}(x,y,z) \neq 0 \Rightarrow Dg(x,y)$?

$\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$ = ?

$\uparrow$ $\uparrow$

$F(\dot{x}, y, z(\dot{x}, y)) = 0$

$$h(x,y) = F(x, y, z(x,y)) = 0$$

Regra da cadeia:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \boxed{\frac{\partial z}{\partial x}} = 0$$

$$\frac{\partial F}{\partial x}\big(x,y,z(x,y)\big) + \underbrace{\boxed{\frac{\partial F}{\partial z}(x,y,z(x,y))}}_{\neq 0} \underbrace{\boxed{\frac{\partial z}{\partial x}(x,y)}}_{?} = 0$$

$$\frac{\partial z}{\partial x}(x,y) = \frac{\partial g}{\partial x}(x,y) = -\frac{\frac{\partial F}{\partial x}(x,y,z(x,y))}{\frac{\partial F}{\partial z}(x,y,z(x,y))}$$

...

$$h(x,y) = m(x,y)$$

$$\frac{\partial h}{\partial x} = \frac{\partial m}{\partial x}$$

$$h(x,y) - m(x,y) = 0$$

$$\frac{\partial h}{\partial x} - \frac{\partial m}{\partial x} = 0$$

6- (substituto)

$$\frac{\partial}{\partial x} g\Big(\underbrace{g(\overset{a}{x^2}, \overset{b}{xy}, \overset{c}{x+y})}_{u(x,y)}, \underbrace{xy}_{v(x,y)}, \underbrace{x+y}_{w(x,y)}\Big)$$

$$\rightarrow g\left(u(x,y), v(x,y), w(x,y)\right)$$

$$u(x,y) = g\left(x^2, xy, x+y\right)$$

$$u(x,y) = g\left(a(x,y), b(x,y), c(x,y)\right) \leftarrow$$

$$\frac{\partial g}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\left(\frac{\partial v}{\partial x}\right)^{y} + \frac{\partial g}{\partial w}\left(\frac{\partial w}{\partial x}\right)^{1}$$

$$\frac{\partial u}{\partial x} = \frac{\partial g}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial g}{\partial b}\frac{\partial b}{\partial x} + \frac{\partial g}{\partial c}\frac{\partial c}{\partial x}$$

$$g(x,y,z) \qquad g(a,b,c) \qquad g(u,v,w)$$

$$g : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$h(x,y) = g\big(g(x,y),\, y\big) \qquad g(1,2) = 3$$

$$= g\big(u(x,y),\, v(x,y)\big)$$

$$\underbrace{(1,2)}_{}\overset{x,y}{} \longrightarrow \underbrace{\big(g(1,2),\,2\big)}_{}\overset{u \quad v}{} \longrightarrow g\big(g(1,2),\,2\big)$$

$$\begin{tikzcd}
(1,2) \arrow[r, mapsto] \arrow[rr, bend right, "h"'] & (3,2) \arrow[r] & g(3,2)
\end{tikzcd}$$

$$\frac{\partial h}{\partial x}(1,2) = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x}$$

$(3,2)$ $\qquad\qquad$ $(1,2)$

$$u(x,y) = g(x,y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial g}{\partial x}$$